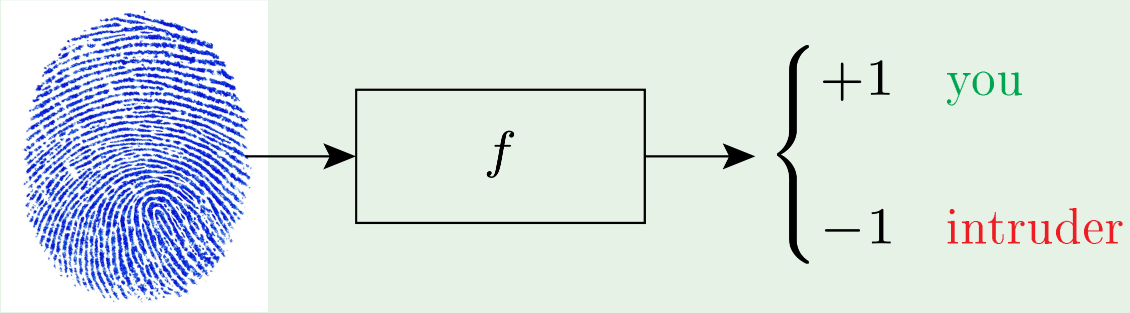
Review of Lecture 4

- Error measures

  - User-specified $\mathrm{e}\left(h(\mathbf{x}), f(\mathbf{x})\right)$



  - In-sample:

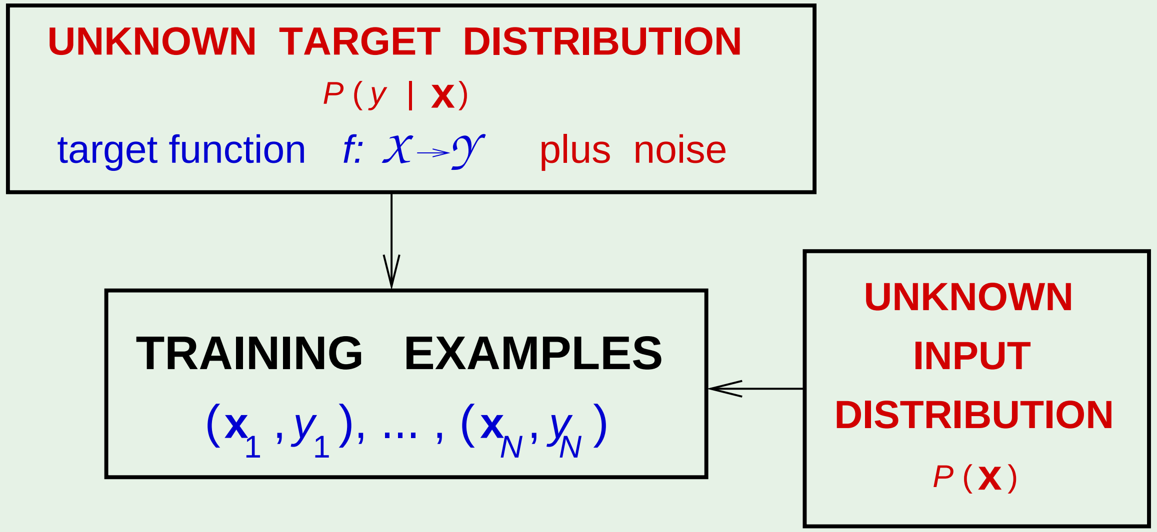
$$E_{\text{in}}(h) = \frac{1}{N}\sum_{n=1}^{N} \mathrm{e}\left(h(\mathbf{x}_n), f(\mathbf{x}_n)\right)$$

  - Out-of-sample

$$E_{\text{out}}(h) = \mathbb{E}_{\mathbf{x}}\left[\mathrm{e}\left(h(\mathbf{x}), f(\mathbf{x})\right)\right]$$

- Noisy targets

$$y = f(\mathbf{x}) \quad \longrightarrow \quad y \sim P(y \mid \mathbf{x})$$



  - $(\mathbf{x}_1, y_1), \cdots, (\mathbf{x}_N, y_N)$ generated by

$$P(\mathbf{x}, y) = P(\mathbf{x})P(y|\mathbf{x})$$

  - $E_{\text{out}}(h)$ is now $\mathbb{E}_{\mathbf{x},y}\left[\mathrm{e}\left(h(\mathbf{x}), y\right)\right]$

Learning From Data

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Lecture 5: Training versus Testing

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Outline

- From training to testing
- Illustrative examples
- Key notion: break point
- Puzzle

The final exam

Testing:

$$\mathbb{P}\left[\,|E_{\text{in}} - E_{\text{out}}| > \epsilon\,\right] \le \quad 2 \;\; e^{-2\epsilon^2 N}$$

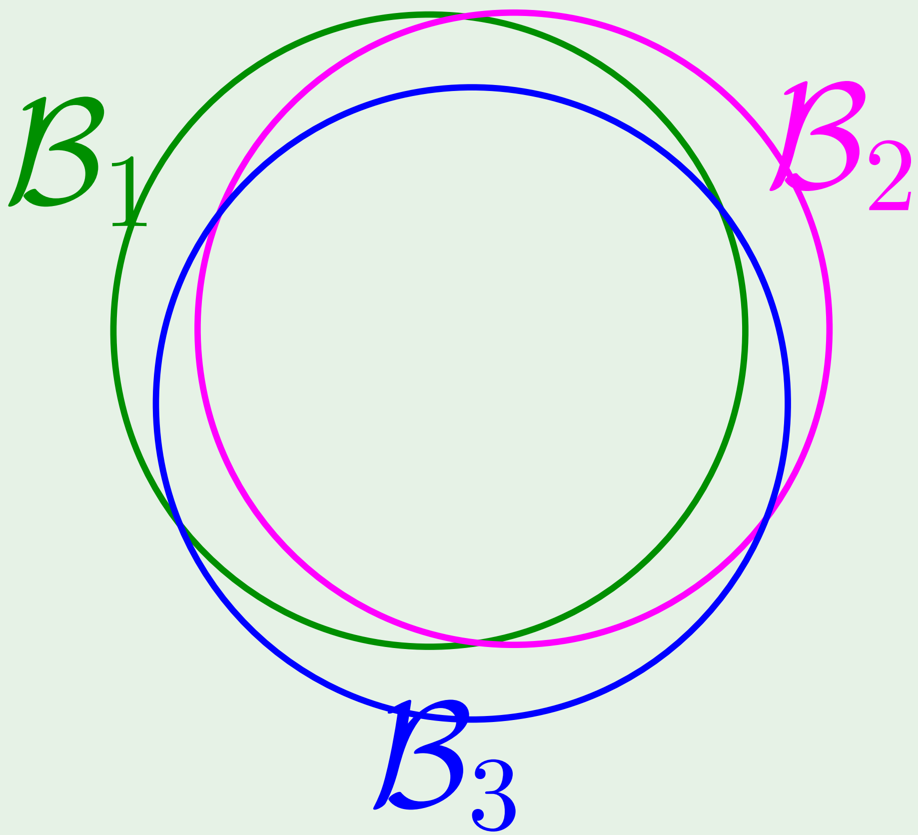
Training:

$$\mathbb{P}\left[\,|E_{\text{in}} - E_{\text{out}}| > \epsilon\,\right] \le 2{\color{red}M}\,e^{-2\epsilon^2 N}$$

Where did the M come from?

The $\mathcal{B}$ad events $\mathcal{B}_m$ are

$$\text{“}|E_{\text{in}}(h_m) - E_{\text{out}}(h_m)| > \epsilon\text{”}$$



The union bound:

$$\mathbb{P}[\mathcal{B}_1 \textbf{ or } \mathcal{B}_2 \textbf{ or } \cdots \textbf{ or } \mathcal{B}_M]$$

$$\leq \underbrace{\mathbb{P}[\mathcal{B}_1] + \mathbb{P}[\mathcal{B}_2] + \cdots + \mathbb{P}[\mathcal{B}_M]}_{\text{no overlaps: } M \text{ terms}}$$

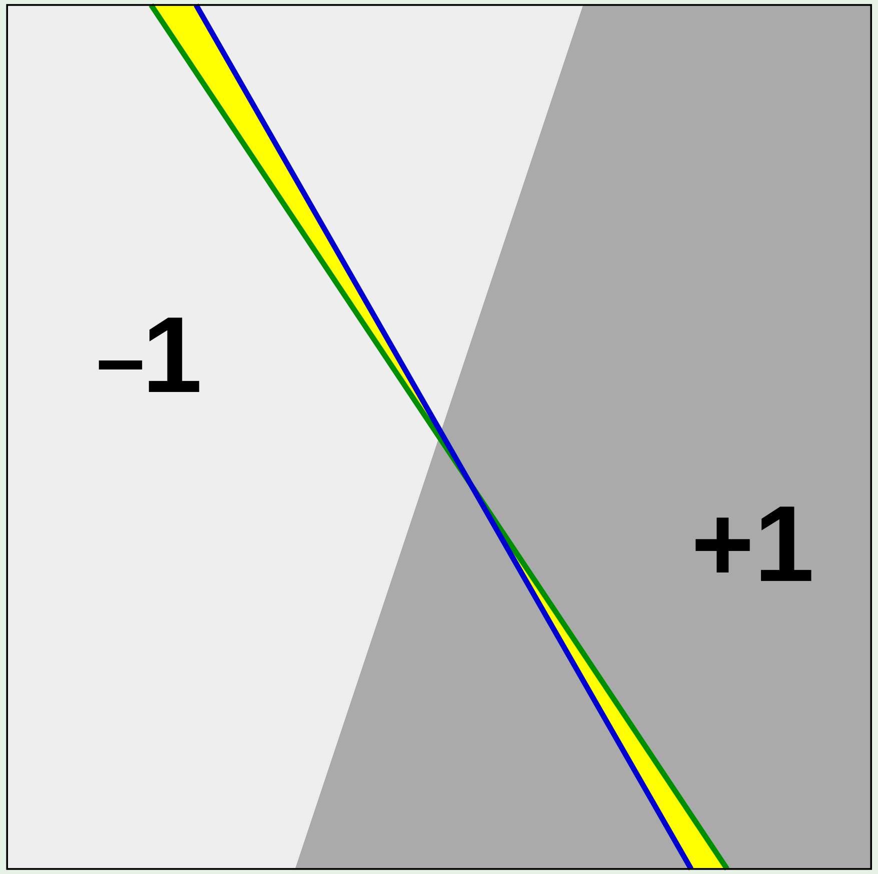
Can we improve on M?

Yes, bad events are *very* overlapping!

ΔE_{out}: change in $+1$ and -1 areas

ΔE_{in}: change in labels of data points

$$|E_{\text{in}}(h_1) - E_{\text{out}}(h_1)| \approx |E_{\text{in}}(h_2) - E_{\text{out}}(h_2)|$$

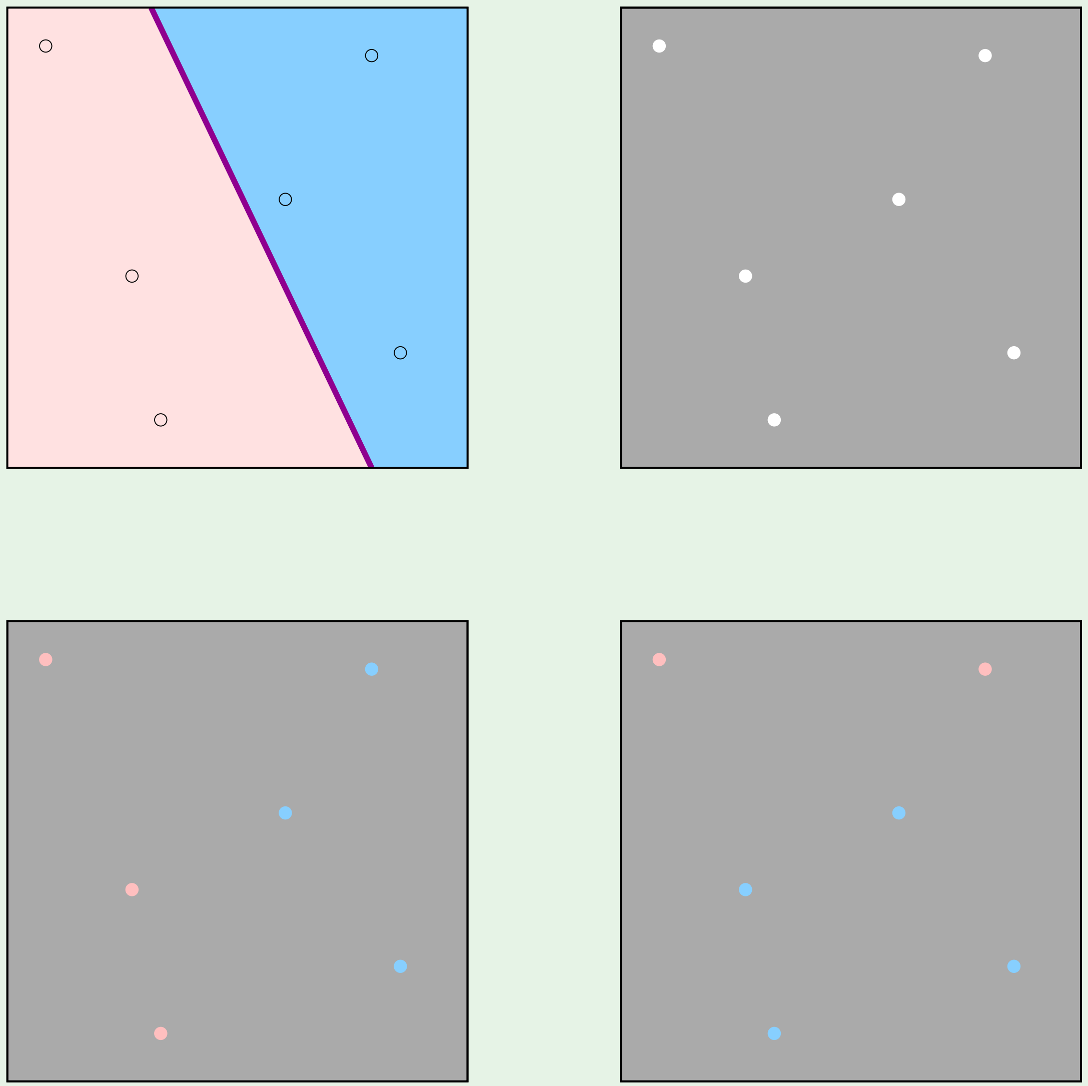


What can we replace M with?

Instead of the whole input space,

we consider a finite set of input points,

and count the number of *dichotomies*



Dichotomies: mini-hypotheses

A hypothesis $h : \mathcal{X} \rightarrow \{-1, +1\}$

A dichotomy $h : \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\} \rightarrow \{-1, +1\}$

Number of hypotheses $|\mathcal{H}|$ can be infinite

Number of dichotomies $|\mathcal{H}(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N)|$ is at most 2^N

Candidate for replacing M

The growth function

The growth function counts the most dichotomies on any N points

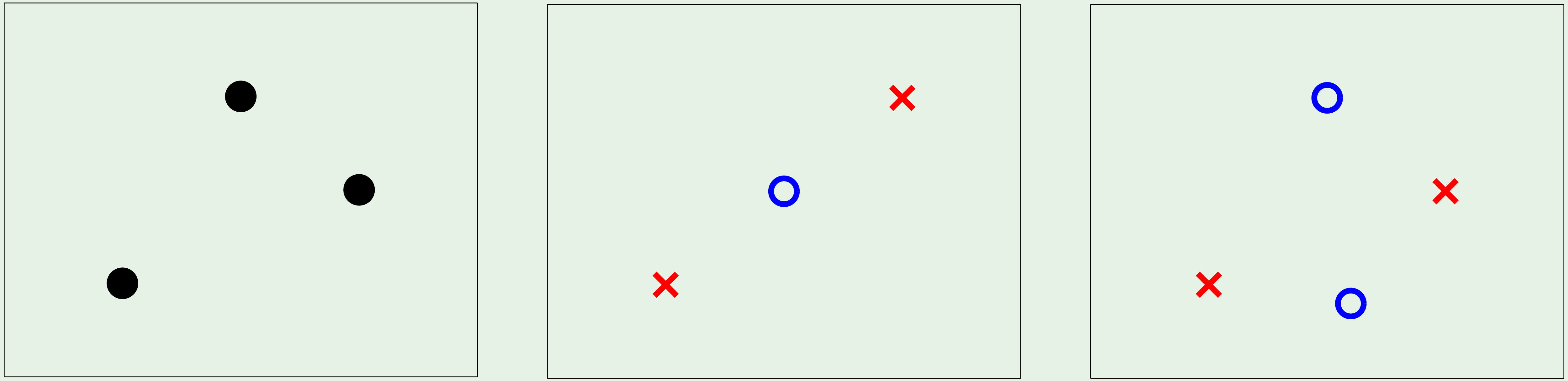
$$m_{\mathcal{H}}(N) = \max_{\mathbf{x}_1, \cdots, \mathbf{x}_N \in \mathcal{X}} |\mathcal{H}(\mathbf{x}_1, \cdots, \mathbf{x}_N)|$$

The growth function satisfies:

$$m_{\mathcal{H}}(N) \leq 2^N$$

Let's apply the definition.

Applying $m_{\mathcal{H}}(N)$ definition - perceptrons



$N = 3$ $\quad$ $N = 3$ $\quad$ $N = 4$

$$m_{\mathcal{H}}(3) = 8 \qquad\qquad m_{\mathcal{H}}(4) = 14$$

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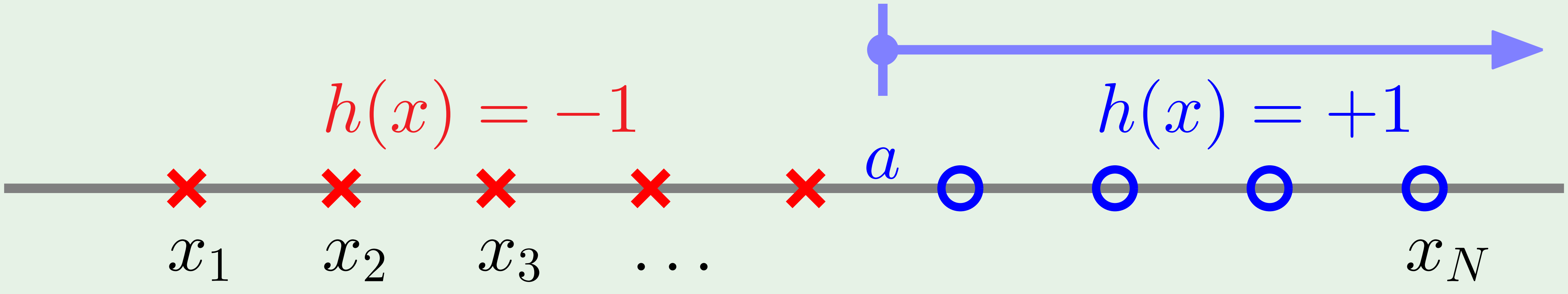
Example 1: positive rays

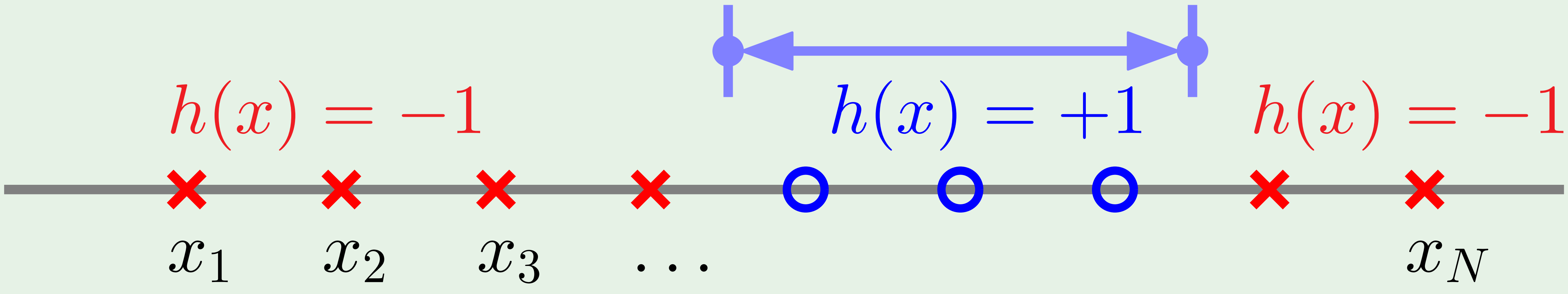


$\mathcal{H}$ is set of $h : \mathbb{R} \to \{-1, +1\}$

$h(x) = \text{sign}(x - a)$

$m_{\mathcal{H}}(N) = N + 1$

Example 2: positive intervals



$\mathcal{H}$ is set of $h \colon \mathbb{R} \to \{-1, +1\}$

Place interval ends in two of $N+1$ spots

$$m_{\mathcal{H}}(N) = \binom{N+1}{2} + 1 = \frac{1}{2}N^2 + \frac{1}{2}N + 1$$

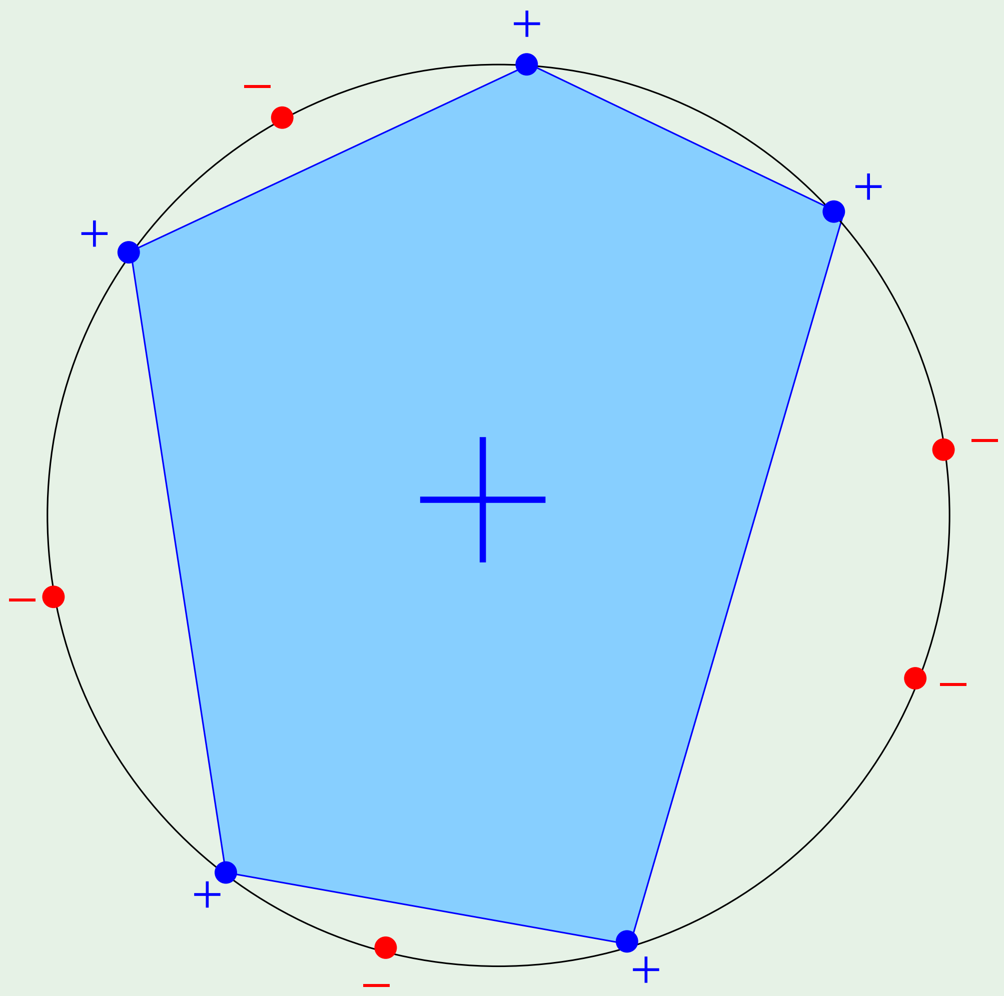
Example 3: convex sets

$\mathcal{H}$ is set of $h\colon \mathbb{R}^2 \to \{-1,+1\}$

$h(\mathbf{x}) = +1$ is convex

$m_{\mathcal{H}}(N) = 2^N$

The N points are 'shattered' by convex sets



The 3 growth functions

- $\mathcal{H}$ is positive rays:

$$m_{\mathcal{H}}(N) = N + 1$$

- $\mathcal{H}$ is positive intervals:

$$m_{\mathcal{H}}(N) = \tfrac{1}{2}N^2 + \tfrac{1}{2}N + 1$$

- $\mathcal{H}$ is convex sets:

$$m_{\mathcal{H}}(N) = 2^N$$

Back to the big picture

Remember this inequality?

$$\mathbb{P}\left[\,|E_{\text{in}} - E_{\text{out}}| > \epsilon\,\right] \le 2M e^{-2\epsilon^2 N}$$

What happens if $m_{\mathcal{H}}(N)$ replaces M?

$m_{\mathcal{H}}(N)$ polynomial $\quad\Longrightarrow\quad$ Good!

Just prove that $m_{\mathcal{H}}(N)$ is polynomial?

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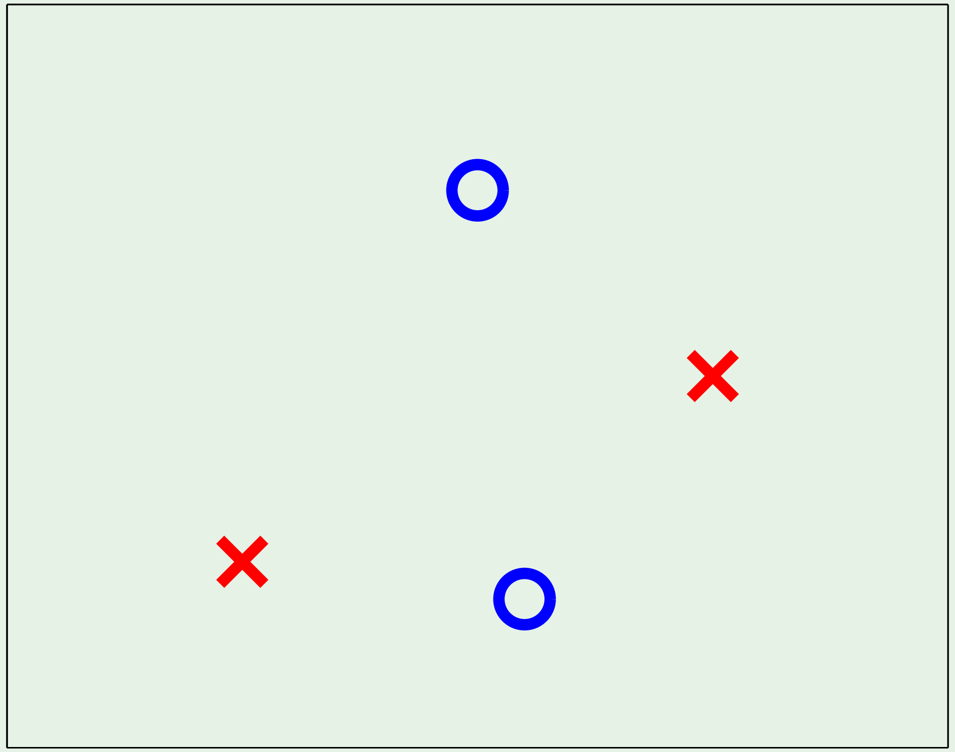
Break point of $\mathcal{H}$

Definition:

If no data set of size k can be shattered by $\mathcal{H}$, then k is a *break point* for $\mathcal{H}$

$$m_{\mathcal{H}}(k) < 2^k$$

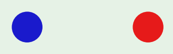


For 2D perceptrons, $k = 4$

A bigger data set cannot be shattered either

Break point - the 3 examples

- Positive rays $m_{\mathcal{H}}(N) = N + 1$

break point $k =$ 2 

- Positive intervals $m_{\mathcal{H}}(N) = \frac{1}{2}N^2 + \frac{1}{2}N + 1$

break point $k =$ 3 

- Convex sets $m_{\mathcal{H}}(N) = 2^N$

break point $k =$ '∞'

Main result

No break point $\Longrightarrow$ $m_{\mathcal{H}}(N) = 2^N$

Any break point $\Longrightarrow$ $m_{\mathcal{H}}(N)$ is **polynomial** in N

Puzzle

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
○	○	○
○	○	●
○	●	○
●	○	○