Homework Assignment 6, Solutions

Problem 1.

For LDPC codes we have: $R = 1 - \dfrac{1}{a\sum_{i=2}^{N}\dfrac{\lambda_i}{i}}$. Here $R = \dfrac{4}{5}$. So we get:

$a = \dfrac{5}{\sum_{i=2}^{N}\dfrac{\lambda_i}{i}} \geq \dfrac{5}{\sum_{i=2}^{N}\dfrac{\lambda_i}{2}} = \dfrac{10}{\sum_{i=2}^{N}\lambda_i} = 10$. So the smallest possible value for a is 10, which is achieved when $N = 2$ and $\lambda_2 = 1$.

Problem 2.

(a) Let $p_1, p_2, \cdots, p_n$ be such a probability density: $\sum_{i=1}^{n} p_i H_i = U$.

Let $q_1, q_2, \cdots, q_n$ be such a probability density: $q_i = \dfrac{e^{-\beta H_i}}{Z(\beta)}$ for $i = 1,2,\cdots,n$; $\sum_{i=1}^{n} q_i H_i = U$. Here $Z(\beta) = \sum_{i=1}^{n} e^{-\beta H_i}$.

$\log x$ is a concave function. So by Jensen's inequality, we have:

$$\sum_{i=1}^{n} p_i \log\frac{q_i}{p_i} \leq \log\left(\sum_{i=1}^{n} p_i \cdot \frac{q_i}{p_i}\right) = \log\left(\sum_{i=1}^{n} q_i\right) = \log 1 = 0.$$

$$\therefore \sum_{i=1}^{n} p_i \log q_i - \sum_{i=1}^{n} p_i \log p_i \leq 0$$

$$\Rightarrow -\sum_{i=1}^{n} p_i \log p_i \leq -\sum_{i=1}^{n} p_i \log q_i = -\sum_{i=1}^{n} p_i \log\frac{e^{-\beta H_i}}{Z(\beta)} = -\sum_{i=1}^{n} p_i \log e^{-\beta H_i} + \sum_{i=1}^{n} p_i \log Z(\beta)$$

$$= \sum_{i=1}^{n} p_i \cdot \beta H_i + \log Z(\beta) \sum_{i=1}^{n} p_i = \beta U + \log Z(\beta) \qquad \cdots\cdots\cdots\cdots (1)$$

Equality in (1) holds when $p_i = q_i$ for $i = 1,2,\cdots,n$. Therefore the optimizing probabilities are of the form: $p_i = \dfrac{e^{-\beta H_i}}{Z(\beta)}$ for $i = 1,2,\cdots,n$, where $Z(\beta) = \sum_{i=1}^{n} e^{-\beta H_i}$.

Q.E.D.

(b) $$-\frac{d}{d\beta}\log Z(\beta) = -\frac{1}{Z(\beta)}\frac{d}{d\beta}Z(\beta) = -\frac{1}{Z(\beta)}\frac{d}{d\beta}\left(\sum_{i=1}^{n} e^{-\beta H_i}\right) = \frac{1}{Z(\beta)}\sum_{i=1}^{n} H_i e^{-\beta H_i}$$

$$= \sum_{i=1}^{n} \frac{e^{-\beta H_i}}{Z(\beta)} H_i = \sum_{i=1}^{n} q_i H_i = U.$$

By inequality (1) in part (a) we know: $S_{\max} = \beta U + \log Z(\beta)$. So:

$S_{\max} = \log Z(\beta) - \beta \frac{d}{d\beta} \log Z(\beta)$. Q.E.D.

(c) $(H_1, H_2, H_3) = (1,2,3)$. So $U = \sum_{i=1}^{3} q_i H_i = \frac{e^{-\beta} + 2e^{-2\beta} + 3e^{-3\beta}}{e^{-\beta} + e^{-2\beta} + e^{-3\beta}}$, and $S_{\max} = \log Z(\beta) + \beta U = \log(e^{-\beta} + e^{-2\beta} + e^{-3\beta}) + \beta U$. Let β run from $-\infty$ to $+\infty$, we get the following plot (the *x* axis is U, and the *y* axis is $S_{\max}$):