EE/Ma 127c Error-Correcting Codes Anxiao (Andrew) Jiang
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Homework Assignment 3, Solutions

Problem 1.

$\Pr\{X_i = j \mid \underline{Y} = \underline{y}^e\} = \Pr\{X_i = j, Y_1 = 1, Y_2 = 2, Y_3 = 2, Y_4 = 0\} / \Pr\{Y_1 = 1, Y_2 = 2, Y_3 = 2, Y_4 = 0\}$

Define $K \overset{\Delta}{=} 1/\Pr\{Y_1 = 1, Y_2 = 2, Y_3 = 2, Y_4 = 0\}$, then

$$\Pr\{X_i = j \mid \underline{Y} = \underline{y}^e\} = K \sum_{\underline{X}: X_i = j} \Pr\{X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4\} = K \sum_{\underline{X}: X_i = j} \prod_{k=1}^{4} \Pr\{X_k \mid X_{k-1}\} \Pr\{Y_k = y_k^e \mid X_k\}.$$

One way to compute is by using the BCJR algorithm. And notice that for any *i*,

$$1/K = \Pr\{Y_1 = 1, Y_2 = 2, Y_3 = 2, Y_4 = 0\} = \sum_{j=0}^{2} \Pr\{X_i = j, Y_1 = 1, Y_2 = 2, Y_3 = 2, Y_4 = 0\}.$$

Below we give the values of $\Pr\{X_i = j \mid \underline{Y} = \underline{y}^e\}$ for $i = 1,2,3,4$ and $j = 0,1,2$:

$\Pr\{X_i = j \mid \underline{Y} = \underline{y}^e\}$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$j = 0$	$\frac{3849}{43993} = 0.0875$	$\frac{2679}{43993} = 0.0609$	$\frac{29349}{43993} = 0.6671$	$\frac{38264}{43993} = 0.870$
$j = 1$	$\frac{40144}{43993} = 0.913$	$\frac{3426}{43993} = 0.0779$	$\frac{11380}{43993} = 0.259$	$\frac{3453}{43993} = 0.0785$
$j = 2$	0	$\frac{37888}{43993} = 0.861$	$\frac{3264}{43993} = 0.0742$	$\frac{2276}{43993} = 0.0517$

Problem 2.

(a) Proof: $p(x_2, x_3) = \sum_{x_1, x_4} p(x_1, x_2, x_3, x_4) = \sum_{x_1} \sum_{x_4} p(x_1) p(x_2) p(x_3 \mid x_1) p(x_4 \mid x_1, x_2)$

$= p(x_2) \sum_{x_1} p(x_1) p(x_3 \mid x_1) \sum_{x_4} p(x_4 \mid x_1, x_2) = p(x_2) \sum_{x_1} p(x_1) p(x_3 \mid x_1)$

$= p(x_2) p(x_3).$

(b) The only other independent pair are X_1 and X_2.

(c) For any variable X in the Bayesian network, define a 'source set' of X, denoted by S(X), as: $S(X) = \{Z \mid \text{There exists a path from Z to X}\} \cup \{X\}$.

Then we claim:

$\forall X, Y$ **, X and Y are independent if** $S(X) \cap S(Y) = \varnothing$**.**

The above claim is the same as:

For any two variables X and Y in the Bayesian network, if there is no path from X to Y or from Y to X, and there is no such variable Z that there is a path from Z to X and there is also a path from Z to Y, then X and Y are independent.

We give a proof below.

Proof: Let $V = \{$ All the random variables in the Bayesian network$\}$. Suppose for

$X \in V$ and $Y \in V$, $S(X) \cap S(Y) = \varnothing$, then:

$$p(X,Y) = \sum_{V-\{X,Y\}} p(V) = \sum_{S(X)-\{X\}} \sum_{S(Y)-\{Y\}} \sum_{V-S(X)-S(Y)} p(V)$$

$$= \sum_{S(X)-\{X\}} \sum_{S(Y)-\{Y\}} \sum_{V-S(X)-S(Y)} p(S(X))p(S(Y))p(V-S(X)-S(Y) \mid S(X),S(Y))$$

$$= \sum_{S(X)-\{X\}} p(S(X)) \sum_{S(Y)-\{Y\}} p(S(Y)) \sum_{V-S(X)-S(Y)} p(V-S(X)-S(Y) \mid S(X),S(Y))$$

$$= \sum_{S(X)-\{X\}} p(S(X)) \sum_{S(Y)-\{Y\}} p(S(Y))$$

$$= p(X)p(Y).$$

Therefore X and Y are independent. And that proves our claim.

Problem 3.

(a) pqr multiplications are required to compute the product AB.

(b) For (AB)C, pqr+prs multiplications are required.
For A(BC), qrs+pqs multiplications are required.

(c) pqr+prs=7500, qrs+pqs=75000, so the best way to compute ABC is to parenthesize ABC as (AB)C.