Solutions to Homework Assignment 2

Problem 1.
From the problem, we get the following edge weight matrices:

$$W_1 = [1\ \ 2], W_2 = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 8 & 0 \end{bmatrix}, W_3 = \begin{bmatrix} 4 & 0 & 2 \\ 8 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}, W_4 = \begin{bmatrix} 2 & 0 \\ 4 & 2 \\ 0 & 1 \end{bmatrix}, W_5 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(a) The α_i's are defined as $\alpha_0 = 1, \alpha_i = \alpha_{i-1} W_i$ otherwise. Carrying out these computations gives us

$$\alpha_0 = [1], \alpha_1 = [1, 2], \alpha_2 = [8, 16, 2], \alpha_3 = [160, 18, 56], \alpha_4 = [392, 92], \alpha_5 = [576].$$

Similarly, the β_i's are defined as $\beta_5 = 1, \beta_{i-1} = W_i \beta_i$ otherwise. This gives us

$$\beta_5 = [1]^T, \beta_4 = [1, 2]^T, \beta_3 = [2, 8, 2]^T, \beta_2 = [12, 28, 16]^T, \beta_1 = [80, 248]^T, \beta_0 = [576]^T.$$

Here, A^T denotes the transpose of the matrix A.

(b) Let μ_i be a vector the size of β_i, whose kth element represents the flow through the k depth vertex at the ith stage of the trellis. Then we know that μ_i is given by $\mu_i(k) = \alpha_i(k)\beta_i(k)$. This gives us

$$\mu_0 = [576]^T, \mu_1 = [80, 496]^T, \mu_2 = [96, 448, 32]^T, \mu_3 = [320, 144, 112]^T,$$

$$\mu_4 = [392, 184]^T, \mu_5 = [576]^T.$$

Similarly, let ν_i be a matrix the size of W_i, each of whose elements represents the flow through the edge whose weight is given by the corresponding element of W_i. Then we know that ν_i is given by $\nu_i(j, k) = \alpha_{i-1}(j) W_i(j, k) \beta_i(k)$. This gives us

$$\nu_1 = [80 \;\; 496], \nu_2 = \begin{bmatrix} 48 & 0 & 32 \\ 48 & 448 & 0 \end{bmatrix}, \nu_3 = \begin{bmatrix} 64 & 0 & 32 \\ 256 & 128 & 64 \\ 0 & 16 & 16 \end{bmatrix},$$

$$\nu_4 = \begin{bmatrix} 320 & 0 \\ 72 & 72 \\ 0 & 112 \end{bmatrix}, \nu_5 = \begin{bmatrix} 392 \\ 184 \end{bmatrix}.$$

(c) Denote by α_i' the approximation to $\log_2 \alpha_i$ computed by the algorithm mentioned in this part, and similarly for β_i'. We get the following values for α_i' and β_i'.

$$\alpha_0' = [0], \alpha_1' = [0, 1], \alpha_2' = [2, 4, 1], \alpha_3' = [7, 4, 5], \alpha_4' = [8, 5], \alpha_5' = [8].$$

$$\beta_0' = [8]^T, \beta_1' = [5, 7]^T, \beta_2' = [3, 4, 3]^T, \beta_3' = [1, 2, 1]^T, \beta_4' = [0, 1]^T, \beta_5' = [0]^T.$$

Problem 2.
Let $f(\Delta)$ be the function to be approximated and $g(\Delta)$ the approximating function. Also, let us use the error measure suggested in the problem, i.e.

$$E(f, g) = \sup_{\Delta} |f(\Delta) - g(\Delta)|.$$

Our job is to choose values of $\Delta_1, \Delta_2, \Delta_3$ and y_1, y_2, y_3, y_4 such that this error measure is minimized. Firstly, it is clear that for fixed values of the Δ_i's, the y_i's should be midpoint of the spread of $f(\Delta)$ over the corresponding interval. Since $f(\Delta)$ is a decreasing function, we get

$$y_1 = \frac{f(0) + f(\Delta_1)}{2}, y_2 = \frac{f(\Delta_1) + f(\Delta_2)}{2}, y_3 = \frac{f(\Delta_2) + f(\Delta_3)}{2}, y_4 = \frac{f(\Delta_4) + f(\infty)}{2}.$$

For these choices of the y_i's, the error measure then takes the value equal to half the maximum spread of $f(\Delta)$ over any of these intervals. The total spread of $f(\Delta)$ ever the real line is $\log 2$. Therefore, we should make the spread equal to $\log 2/4$ on each of these intervals to minimize the error measure. Therefore, we get

$$f(\Delta_1) = 3\log 2/4, f(\Delta_2) = \log 2/2, f(\Delta_3) = \log 2/4.$$

From this, we then get

$$y_1 = 7\log 2/8, y_2 = 5\log 2/8, y_3 = 3\log 2/8, y_4 = \log 2/8.$$

Problem 3.

(a) Let us verify the distributive law in each case.
1)Sum-Product: $(a.b) + (a.c) = a.(b + c)$. Usual distributive law.
2)Min-Product: $min(a.b, a.c) = a.min(b, c)$. True if a is positive, which it is.
3)Max-product: $max(a.b, a.c) = a.max(b, c)$. Equivalent to b).
4)Min-Sum: $min(a + b, a + c) = a + min(b, c)$. Always true.
5)Max-Sum: $max(a + b, a + c) = a + max(b, c)$. Similar to d).

(b) To go from min-sum to max-sum and vice versa, you use the $-x$ transformation, since this transformation preserves the sum operation and converts the min to a max. For exactly the same reasons, to go from min-product to max-product, you use the $1/x$ transformation. To convert a sum to a product, you use the e^x transformation. Using these argumants, we end up with the following matrix.

$$\begin{pmatrix} x & 1/x & \log x & -\log x \\ 1/x & x & -\log x & \log x \\ e^x & e^{-x} & x & -x \\ e^{-x} & e^x & -x & x \end{pmatrix}$$

The rows and the columns denote, in order, the semirings min-product, max-product, min-sum and max-sum.