Solutions to Problem Set 4.

Problem 1.
The feasible polytope S is the set of all x satisfying $Ax = b$ and $x \geq 0$. Now consider any $x_1,\ x_2 \in S$ and $z = \alpha x_1 + (1-\alpha)x_2$ with $0 \leq \alpha \leq 1$. Obviously, $z \geq 0$ and

$$Az = A(\alpha x_1 + (1-\alpha)x_2) = \alpha A x_1 + (1-\alpha)Ax_2 = \alpha b + (1-\alpha)b = b$$

so $z \in S$, and thus S is convex.

Problem 2.
Note that the definition given in class seems to give an immediate answer: since the row rank of A is less than m, the number of constraints, then so is the column rank of A, so there cannot be m linearly independent columns of A, and point b) in the definition of a basic feasible point is never satisfied. However technically, the definition was only given for the case when the row rank of A *is* full. A more general definition would be that the submatrix B has to be invertible. But $B \in R^{m \times m}$, so it cannot be invertible for A with rank $< m$. Note that there can nevertheless exist extremal points in this case.

Problem 3.
Assume x^* is primal feasible and y^*, dual feasible with $c^T x^* = b^T y^*$. Suppose x is primal feasible with $c^T x < c^T x^*$. Then by the theorem proved in class, $c^T x < b^T y^*$, which contradicts weak duality. Hence $c^T x \geq c^T x^*$ for all x, and thus x^* is primal optimal.

Suppose y is dual feasible with $b^T y > b^T y^*$. Then by the same theorem, $b^T y > c^T x^*$, which again contradicts weak duality. Hence $b^T y \leq b^T y^*\ \forall y$, and thus y^* is dual optimal.

Problem 4.
Assume that the primal problem has an unbounded objective, but the dual problem has feasible points. Let y be a dual feasible point. Then by weak duality, $b^T y \leq c^T x$ for all feasible points of the primal problem. This implies $b^T y \leq \min\{c^T x\ :\ x \text{ is primal feasible }\}$. But this contradicts the assumption that the primal problem is unbounded. Therefore, the dual problem does not have any feasible points.

Now assume the dual problem is unbounded, but the primal problem has feasible points. Let x be one such point. By weak duality, $b^T y \leq c^T x$ for all dual feasible points y, which implies $\max\{b^T y\ :\ y \text{ is dual feasible }\} \leq c^T x$. But this contradicts the assumption that the dual problem is unbounded. Hence the primal problem cannot have any feasible points.

Problem 5.
Let the column of the constraint matrix A corresponding to the variable x_i' be A_i. The column of A corresponding to x_i'' will then be $-A_i$. Thus the columns corresponding to x_i' and x_i'' are linearly dependent, and any basis that includes both of these columns will be singular. Hence no feasible solution can include both x_i' and x_i'' as basic variables.

Problem 6.

To convert the problem to standard form, introduce slack variables x_5 and x_6. The standard form will then be

$$\begin{aligned} \min\ & -5x_1 - 7x_2 - 12x_3 + x_4 \\ \text{s.t. } & 2x_1 + 3x_2 + 2x_3 + x_4 + x_5 = 38 \\ & 3x_1 + 2x_2 + 4x_3 - x_4 + x_6 = 55 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

Solve this using simplex method:

1st iteration

$$\mathcal{B} = \{x_5, x_6\},\ \mathcal{N} = \{x_1, x_2, x_3, x_4\}$$
$$c_B^T = (0, 0),\ c_N^T = (-5, -7, -12, 1)$$

$$B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad N = \begin{pmatrix} 2 & 3 & 2 & 1 \\ 3 & 2 & 4 & -1 \end{pmatrix}$$

$$x_B^{(1)} = (38, 55)^T,\ y = (0, 0)^T,\ s_N = (-5, -7, -12, 1)^T$$

x_3 is the entering variable, and $t = (2, 4)^T$. The ratios are $\{38/2,\ 55/4\}$, so x_6 is the leaving variable.

2nd iteration

$$\mathcal{B} = \{x_5, x_3\},\ \mathcal{N} = \{x_1, x_2, x_6, x_4\}$$
$$c_B^T = (0, -12),\ c_N^T = (-5, -7, 0, 1)$$

$$B^{-1} = \begin{pmatrix} 1 & -1/2 \\ 0 & 1/4 \end{pmatrix} \qquad N = \begin{pmatrix} 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & -1 \end{pmatrix}$$

$$x_B^{(2)} = (21/2, 55/4)^T,\ y = (0, -3)^T,\ s_N = (4, -1, 3, -2)^T$$

x_4 is the entering variable, and $t = (3/2, -1/4)^T$. The ratios are $\{21/3\}$, so x_5 is the leaving variable.

3rd iteration

$$\mathcal{B} = \{x_4, x_3\},\ \mathcal{N} = \{x_1, x_2, x_6, x_5\}$$
$$c_B^T = (1, -12),\ c_N^T = (-5, -7, 0, 0)$$

$$B^{-1} = \begin{pmatrix} 2/3 & -1/3 \\ 1/6 & 1/6 \end{pmatrix} \qquad N = \begin{pmatrix} 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

$$x_B^{(3)} = (7, 31/2)^T,\ y = (-4/3, -7/3)^T,\ s_N = (14/3, 5/3, 7/3, 4/3)^T$$

This is the optimal basis and the optimal value of the objective is $c_B^T x_B = -179$.

Problem 7.
We are going to prove that if x_p is the leaving variable and x_q is the entering variable, then the reduced cost of the variable x_p becomes positive and thus it cannot re-enter the basis in the following iteration.

Start with $A^T y + s = c$, which can be partitioned into $B^T y + s_B = c_B$ and $N^T y + s_N = = c_N$. From the first expression we get $y = B^{-T}(c_B - s_B)$, while looking at the qth row of the second expression we get $N_q^T y + s_q = A_q^T y + s_q = c_q$. Now substitute the above relation for y: $A_q^T(B^{-T}(c_B - s_B)) + s_q = c_q$. But all the elements in s_B remain zero, except for the one correspoding to the leaving variable p; also note that $A_q^T B^{-T} = (B^{-1}A_q)^T = t^T$. Thus the previous equality yields $t^T c_B - t_p s_p + s_q = c_q$. Therefore $-t_p s_p + s_q = c_q - t^T c_B$. The right-hand side of this expression remains constant in the pricing process; in the left-hand side, s_q becomes zero after the entering variable x_q enters the basis. Hence we conclude that $-t_p s_p = s_q$, or $s_p = -s_q/t_p$. Since $s_q < 0$ and $t_p > 0$, it follows that $s_p > 0$, and so the variable x_p cannot re-enter the basis in the following iteration.

Problem 8.
Let the final Phase I basis be $x_B = \{x_1, \ldots, x_m\}$. Then the nonbasic variables are $x_N = = \{x_{m+1}, \ldots, x_n, a_1, \ldots, a_m\}$. Now $c_B^T = (0, \ldots, 0)$ and $c_N^T = (0, \ldots, 0, 1, \ldots, 1)$. The reduced costs then are $s_N^T = c_N^T - c_B^T B^{-1} N = c_N^T = (0, \ldots, 0, 1, \ldots, 1)$, i.e., the reduced costs of all the original variables are 0 and the reduced costs of all the artificial variables are 1.